Introduction to Poisson processes

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Abstract

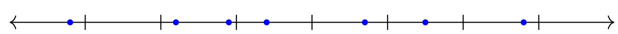
The Poisson process provides a canonical way to build a probability space from a possibly infinite measure space. We give an introduction to Poisson processes from a descriptive set theorist's perspective, with some applications to constructing free pmp actions of Polish groups.

1 Idea

Input: A measure space (X, ρ), usually with X standard Borel and ρ σ-finite.

Output: A probability space $(\mathcal{N}(X), \mathrm{Pois}_X(\rho))$ **canonically** and **faithfully** built from (X, ρ).

Example. For $(X, \rho) = (\mathbb{R}, \lambda)$, a typical sample from $\mathrm{Pois}_{\mathbb{R}}(\lambda)$ is a countable subset of $\mathbb{R}$:



On average, there are $\lambda([n, n+1]) = 1$ points per unit interval.

Theorem (Golodets–Sinel'shchikov 1990, Adams–Elliott–Giordano 1994)**.**
Every locally compact Polish group G admits a free pmp Borel action.

Proof. If G is compact, take Haar measure μ.
Otherwise, take $\mathrm{Pois}_G(\mu)$. □

Remark. If G is countable, we can also take $(2^G, \mu^G)$ (or (X^G, μ^G)) with shift action.
In this case, $\mathrm{Pois}_G(\rho) = (\mathbb{N}^G, \mathrm{Pois}_1(\rho(1))^G)$ (see later Example).

Theorem (Kechris–Malicki–Panagiotopoulos–Zielinski 2022)**.**
If X is a locally compact Polish metric space, then $\mathrm{Iso}(X)$ admits a free pmp action.
In particular, $\mathrm{Iso}(X) \leq \mathrm{Aut}(\mu)$.

Proof. By [Loomis 1949], X admits a regular σ-finite $\mathrm{Iso}(X)$-invariant measure μ.
Take $\mathrm{Pois}_X(\mu)^{\mathbb{N}}$. □

2 Spaces of measures

Definition. Let X be a Borel (i.e., measurable) space.

$$\begin{aligned}
\mathcal{M}(X) &:= \{\mu \in [0,\infty]^{\mathcal{B}(X)} \mid \mu \text{ is a measure}\}, \text{ w/ induced Borel structure} \\
\mathcal{N}(X) &:= \{\nu \in \mathcal{M}(X) \mid \nu \text{ is } \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}\text{-valued}\} \\
\mathcal{M}_{<\infty}(X) &:= \{\mu \in \mathcal{M}(X) \mid \mu(X) < \infty\} \\
\mathcal{M}_1(X) &:= \{\mu \in \mathcal{M}(X) \mid \mu(X) = 1\}
\end{aligned}$$

If X is standard Borel, then so is $\mathcal{M}_1(X)$, as is $\mathcal{M}_{\leq 1}(X)$, hence also $\mathcal{M}_{<\infty}(X) = \bigcup_n \mathcal{M}_{\leq n}(X)$.

Example. A countable subset $N \subseteq X$ can be identified with $\nu \in \mathcal{N}(X)$ given by $\nu(B) := |N \cap B|$.

In general, we can think of $\nu \in \mathcal{N}(X)$ as a "multiset", where $\nu(B)$ is the number of points in "$\nu \cap B$"; the same point may occur multiple times. (Note that there are weird measures in $\mathcal{N}(X)$ which are not a countable sum of Dirac deltas, e.g., $\nu(B) := \infty$ if B is uncountable, else $\nu(B) = 0$. We could restrict to the subspace of countably supported $\nu \in \mathcal{N}(X)$; but below we will restrict to a further subspace $\mathcal{N}_{\mathcal{I}<\infty}(X)$ which is moreover standard Borel. For maximum generality, we will begin by working on the full non-standard Borel space $\mathcal{N}(X)$.)

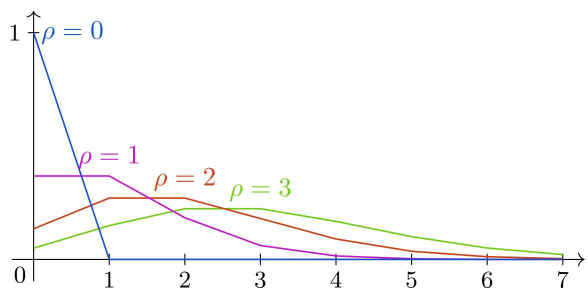
3 Poisson measures

The Poisson process construction takes $\mathcal{M}(X) \ni \rho \longmapsto \mathrm{Pois}_X(\rho) \in \mathcal{M}_1(\mathcal{N}(X))$.

For $X = 1$. Given $\rho \in \mathcal{M}(1) \cong [0, \infty]$, we define $\mathrm{Pois}(\rho) = \mathrm{Pois}_1(\rho) \in \mathcal{M}_1(\mathcal{N}(1)) \cong \mathcal{M}_1(\overline{\mathbb{N}})$ by

$$\mathrm{Pois}(\rho)(n) := e^{-\rho}\frac{\rho^n}{n!}$$

for $\rho < \infty$ and $n \in \mathbb{N}$ (with $\mathrm{Pois}(\rho)(\infty) = 0$).



For $\rho = \infty$, by convention $\mathrm{Pois}(\infty) := \delta_\infty \in \mathcal{M}_1(\overline{\mathbb{N}})$.

Proposition. $\mathrm{Pois}(\sigma + \tau) = \mathrm{Pois}(\sigma) * \mathrm{Pois}(\tau) := +_*(\mathrm{Pois}(\sigma) \times \mathrm{Pois}(\tau))$, where $+ : \overline{\mathbb{N}}^2 \to \overline{\mathbb{N}}$.

Proof. If $\sigma = \infty$ or $\tau = \infty$, both sides are δ_∞. Otherwise, by the binomial theorem,

$$\begin{aligned}\mathrm{Pois}(\sigma + \tau)(n) = e^{-\sigma-\tau}\frac{(\sigma+\tau)^n}{n!} &= e^{-\sigma-\tau}\sum_{i+j=n}\frac{\sigma^i\tau^j}{i!j!} \\ &= \sum_{i+j=n}\mathrm{Pois}(\sigma)(i)\mathrm{Pois}(\tau)(j) = (\mathrm{Pois}(\sigma) * \mathrm{Pois}(\tau))(n).\end{aligned}$$ □

Note that

$$\begin{array}{ccc} \overline{\mathbb{N}}^2 & \xrightarrow{\;+\;} & \overline{\mathbb{N}} \\ \wr\Vert & & \wr\Vert \\ \mathcal{N}(2) & \xrightarrow{\;f_*\;} & \mathcal{N}(1) \end{array} \qquad \text{where } f : 2 \to 1.$$

Thus the above says: for any $\rho = (\sigma, \tau) \in \mathcal{M}(2) = \overline{\mathbb{N}}^2$,

$$\mathrm{Pois}(f_*(\rho)) = f_{**}(\underbrace{\mathrm{Pois}(\sigma) \times \mathrm{Pois}(\tau)}_{=:\mathrm{Pois}_2(\rho)}).$$

An easy induction/approximation argument shows the obvious generalization to $\rho \in \overline{\mathbb{N}}^n$, $n \leq \omega$.

For countable X**.** Given $\rho \in \mathcal{M}(X) \cong [0, \infty]^X$,

$$\mathrm{Pois}_X(\rho) := \prod_{x \in X} \mathrm{Pois}(\rho(x)) \in \mathcal{M}_1(\overline{\mathbb{N}}^X) \cong \mathcal{M}_1(\mathcal{N}(X)).$$

The above Proposition, generalized to countable arity, now says that for $f : X \to 1$,

$$\mathrm{Pois}_1(f_*(\rho)) = f_{**}(\mathrm{Pois}_X(\rho)).$$

By taking a countable disjoint union, this also holds for $f : X \to Y$ between two countable sets.

$$\begin{array}{ccccc} X & \qquad & \mathcal{M}(X) & \xrightarrow{\mathrm{Pois}_X} & \mathcal{M}_1(\mathcal{N}(X)) \\ {\scriptstyle f}\downarrow & & {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_{**}} \\ Y & & \mathcal{M}(Y) & \xrightarrow{\mathrm{Pois}_Y} & \mathcal{M}_1(\mathcal{N}(Y)) \end{array}$$

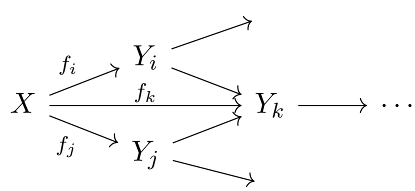
Example. Let G be a countable group, $\rho \in \mathcal{M}(G)$ be counting measure.
Each $g : G \to G$ preserves ρ, i.e., $g_*(\rho) = \rho$, whence

$$\mathrm{Pois}_G(\rho) = \mathrm{Pois}_G(g_*(\rho)) = g_{**}(\mathrm{Pois}_G(\rho)).$$

So we have a pmp $G \curvearrowright (\mathcal{N}(G), \mathrm{Pois}_G(\rho)) = (\overline{\mathbb{N}}^G, \mathrm{Pois}(1)^G)$, where

$$\mathrm{Pois}(1) = (1/e, 1/e, 1/2e, 1/6e, \dots) \quad \text{(supported on } \mathbb{N}).$$

For general standard Borel X**.** Let $\rho \in \mathcal{M}(X)$. Consider all Borel "approximations" $f : X \to Y$ by countable sets Y. In other words, we are looking at countable Borel partitions $X = \bigsqcup_{y \in Y} f^{-1}(y)$.



Note that this is an *uncountable* inverse system. It is nonetheless easily seen that

$$X \cong \varprojlim_{f : X \to Y} Y,$$
$$\mathcal{N}(X) \cong \varprojlim_{f : X \to Y} \mathcal{N}(Y).$$

The **Poisson process** $\mathrm{Pois}_X(\rho) \in \mathcal{M}_1(\mathcal{N}(X))$ is the unique measure such that for each $f : X \to Y$,

$$\mathrm{Pois}_Y(f_*(\rho)) = f_{**}(\mathrm{Pois}_X(\rho)).$$

In other words, if we pick a $\mathrm{Pois}_X(\rho)$-random multiset $\nu \in \mathcal{N}(X)$, then for any countable Borel partition $f : X \to Y$, the collection of numbers $(\nu(f^{-1}(y)))_{y \in Y} = f_*(\nu) \in \mathcal{N}(Y) = \overline{\mathbb{N}}^Y$ will be distributed according to $\mathrm{Pois}_Y(f_*(\rho)) = \prod_{y \in Y} \mathrm{Pois}(\rho(f^{-1}(y)))$.

Proposition. If ρ is σ-finite, then such a measure exists (and is automatically unique).
(Existence requires proof, because the above inverse system is uncountable; otherwise we could just apply the Kolmogorov consistency theorem.)

Remark. Even assuming ρ is σ-finite, $\mathrm{Pois}_X(\rho)$ by default lives on the non-standard Borel $\mathcal{N}(X)$. However, for each countable Borel partition $f : X \to Y$ such that each $\rho(f^{-1}(y)) < \infty$, we have

$$\begin{aligned}\mathrm{Pois}_X(\rho)\big(\{\nu \in \mathcal{N}(X) \mid \forall y\,(\underbrace{\nu(f^{-1}(y))}_{f_*(\nu)(y)} < \infty)\}\big) &= f_{**}(\mathrm{Pois}_X(\rho))(\mathbb{N}^Y \subseteq \overline{\mathbb{N}}^Y = \mathcal{N}(Y)) \\ &= \mathrm{Pois}_Y(f_*(\rho))(\mathbb{N}^Y) = \textstyle\prod_{y\in Y} \mathrm{Pois}(\rho(f^{-1}(y)))(\mathbb{N}) = 1;\end{aligned}$$

thus $\mathrm{Pois}_X(\rho)$ lives on the standard Borel subspace

$$\mathcal{N}_{\{f^{-1}(y)\}_y<\infty}(X) := \{\nu \in \mathcal{N}(X) \mid \forall y\,(\nu(f^{-1}(y)) < \infty)\} \cong \prod_{y\in Y} \mathcal{N}_{<\infty}(f^{-1}(y)).$$

In most concrete situations, there will not be a canonical such partition. However, note that this space is the same as $\mathcal{N}_{\mathcal{I}<\infty}(X)$, where $\mathcal{I}$ is the ideal generated by the partition. Often, there will be a canonical countably generated such ideal $\mathcal{I}$ with $X = \bigcup \mathcal{I}$ witnessing σ-finiteness of ρ.

Example. For locally compact Polish X, we can take $\mathcal{I}$ to be the ideal of precompact Borel sets; then any locally finite ρ yields $\mathrm{Pois}_X(\rho)$ living on the canonical standard Borel subspace $\mathcal{N}_{\mathcal{I}<\infty}(X) \subseteq \mathcal{N}(X)$ of "locally finite multisets".

Proposition. If ρ is atomless, then $\mathrm{Pois}_X(\rho)$ lives on

$$\mathcal{N}_{\mathsf{set}}(X) := \{\nu \in \mathcal{N}(X) \mid \exists \text{ ctbl } C \subseteq X \text{ s.t. } \nu = \textstyle\sum_{c\in C} \delta_c\}.$$

Proof. Let $f : X \to Y$ be a countable Borel partition. Then $\mathrm{Pois}_X(\rho)(\mathcal{N}_{\mathsf{set}}(X))$ is at least

$$\begin{aligned}\mathrm{Pois}_X(\rho)\big(\{\nu \in \mathcal{N}(X) \mid \forall y\,(\underbrace{\nu(f^{-1}(y))}_{f_*(\nu)(y)} \le 1)\}\big) &= f_{**}(\mathrm{Pois}_X(\rho))(\{0,1\}^Y \subseteq \overline{\mathbb{N}}^Y = \mathcal{N}(Y)) \\ &= \mathrm{Pois}_Y(f_*(\rho))(\{0,1\}^Y) \\ &= \prod_{y\in Y} \mathrm{Pois}(\rho(f^{-1}(y)))(\{0,1\}) \\ &= \prod_{y\in Y} e^{-\rho(f^{-1}(y))}\big(1 + \rho(f^{-1}(y))\big) \\ &= \exp\left(\sum_{y\in Y}\Big(-\rho(f^{-1}(y)) + \log\big(1+\rho(f^{-1}(y))\big)\Big)\right) \\ &\ge \exp\left(-\sum_{y\in Y} \rho(f^{-1}(y))^2/2\right).\end{aligned}$$

Since ρ is atomless, we can choose f so that $\sum_y \rho(f^{-1}(y))^2 \to 0$. $\square$

Proposition. For any Borel automorphism $T : X \to X$ such that $T_*\rho = \rho$, we have

$$\mathrm{Pois}_X(\rho)(\{\nu \in \mathcal{N}(X) \mid T_*\nu \ne \nu\}) \ge 1 - \|\mathrm{Pois}(\rho(\{x \in X \mid Tx \ne x\}))\|_2$$

where $r \mapsto 1 - \|\mathrm{Pois}(r)\|_2 : [0,\infty] \to [0,1]$ is an order-isomorphism.

In particular, if T's set of non-fixed points had infinite ρ-measure, then T_* is free $\mathrm{Pois}_X(\rho)$-a.e.